3.2 - Nonlinear Models

Logistic growth: $\dfrac{dP}{dt} = P(a - bP)$

(For large populations or when environmental factors inhibit growth)

Note: $-bP^2$ $(b>0)$ is an <u>inhibition</u> or <u>competition</u> term

Justification for the model:

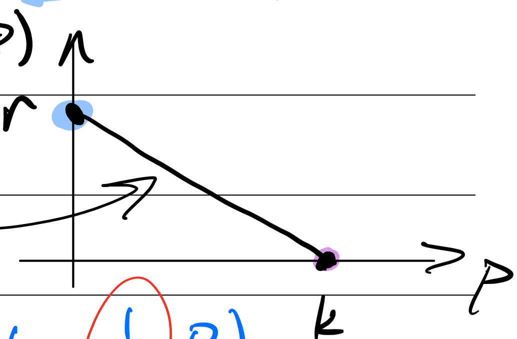
$\dfrac{dP/dt}{P} = k$ - constant relative growth rate (3.1)

$\dfrac{dP/dt}{P} = f(P)$ - variable growth rate but still depends on the population

If K is the carrying capacity for the population (max population)

$f(K) = 0$. If we let $f(0) = r$, then in the simplest case,

$f(P) = r - \dfrac{r}{K}P$ linear



Then $\dfrac{dP}{dt} = P\left(r - \dfrac{r}{K}P\right) = rP\left(1 - \dfrac{1}{K}P\right)$

Relabeling gives $\dfrac{dP}{dt} = P(a - bP) = aP\left(1 - \dfrac{b}{a}P\right)$

$K = \dfrac{a}{b}$ - carrying capacity

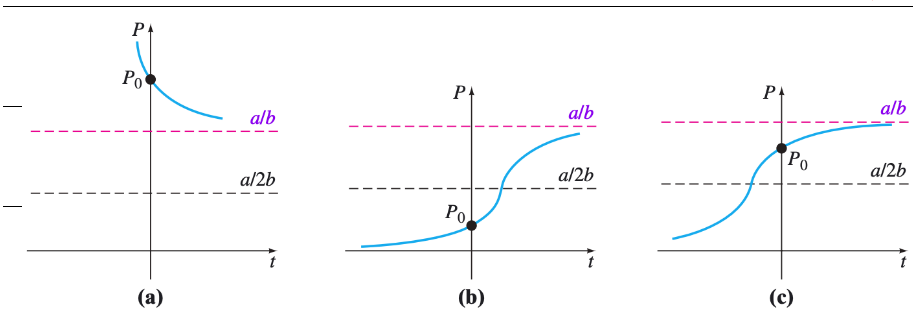


FIGURE 3.2.2 Logistic curves for different initial conditions

rate of restocking

Modified logistic growth: $\frac{dP}{dt} = P(a - bP) \pm h$

rate of harvesting

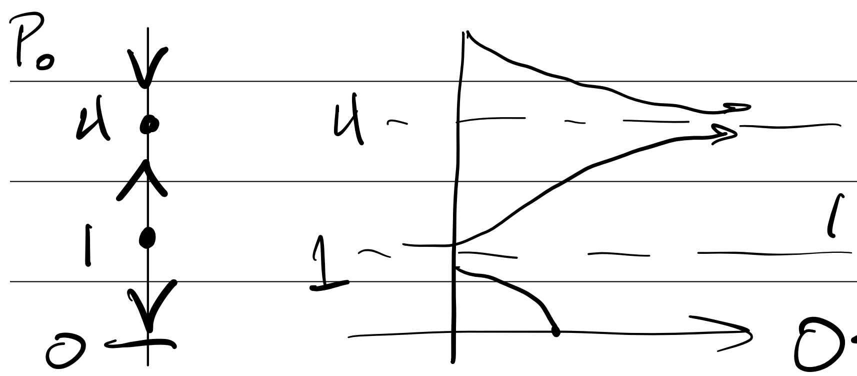
Example: (a) If a constant number h of fish are harvested from a fishery per unit time, then a model for the population $P(t)$ of the fishery at time t is given by $\frac{dP}{dt} = P(a - bP) - h$, $P(0) = P_0$, where a, b, h, and P_0 are positive constants. Suppose $a = 5$, $b = 1$, and $h = 4$. Since the DE is autonomous, use the phase portrait concept of Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. Determine the long-term behavior of the population in each case.

$-(P-4)(P-1)$

$\frac{dP}{dt} = P(5-P) - 4 = -P^2 + 5P - 4 = -(P^2 - 5P + 4)$

$\frac{dP}{dt} = 0 \Rightarrow P = 1, 4$ critical points

phase portrait



$P_0 > 4$ – pop will decline to $P = 4$

$1 < P_0 < 4$ – pop will increase to $P = 4$

$0 < P_0 < 1$ – pop will go extinct.

(b) Solve the IVP in part (a). Verify the results of your phase portrait in part (a) by using a graphing utility to plot the graph of $P(t)$ with an initial condition taken from each of the three intervals given.

$\frac{dP}{dt} = P(5-P) - 4 = -(P^2 - 5P + 4) \Rightarrow \frac{dP}{(P-4)(P-1)} = -dt$

$\frac{1}{(P-4)(P-1)} = \frac{A}{P-4} + \frac{B}{P-1}$ $\quad 1 = A(P-1) + B(P-4)$

$A = \frac{1}{3}, \; B = -\frac{1}{3}$

$\int \left(\frac{1/3}{P-4} - \frac{1/3}{P-1} \right) dP = -t + C_1$

$$\frac{1}{3}\ln\left|\frac{P-4}{P-1}\right| = -t + C_1 \Rightarrow \ln\left|\frac{P-4}{P-1}\right| = -3t + C_2$$

$$\ln\left|\frac{P-1}{P-4}\right| = 3t + C_3$$

$$\frac{P-1}{P-4} = ce^{3t} \qquad P(0) = P_0$$

$$\boxed{c = \frac{P_0 - 1}{P_0 - 4}}$$ (we'll use "c" for now)

$$P - 1 = (P-4)ce^{3t}$$

$$P - 1 = Pce^{3t} - 4ce^{3t}$$

$$P(1 - ce^{3t}) = 1 - 4ce^{3t}$$

$$P = \frac{1 - 4ce^{3t}}{1 - ce^{3t}}$$

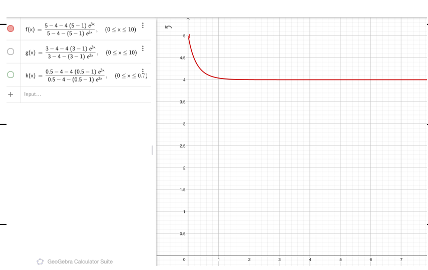
$$P = \frac{e^{-3t} - 4\frac{P_0-1}{P_0-4}}{e^{-3t} - \frac{P_0-1}{P_0-4}}$$

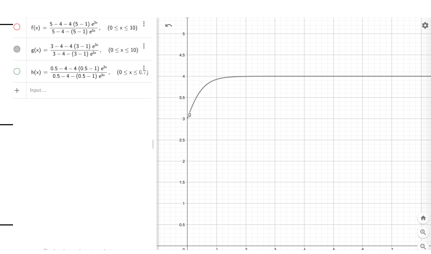
$$\boxed{P(t) = \frac{4(P_0-1) - (P_0-4)e^{-3t}}{P_0 - 1 - (P_0-4)e^{-3t}}}$$

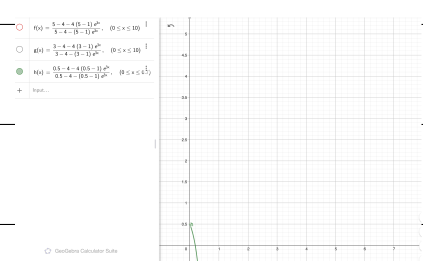
$0 < P_0 < 1$

$1 < P_0 < 4$

$P_0 > 4$







(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

$P=0 \Rightarrow$ numerator $=0$

we find $\quad t = -\frac{1}{3}\ln\left[\frac{4(P_0-1)}{P_0-4}\right]$

Second-order chemical reaction: $\dfrac{dX}{dt} = k(\alpha - X)(\beta - X)$,

where $\alpha = \dfrac{a(M+N)}{M}$ and $\beta = \dfrac{b(M+N)}{N}$,

a = # grams of chemical A,

b = # grams of chemical B, and

$X(t)$ is grams of C, formed from M parts of A and N parts of B

Justification: Amts remaining at time t:

$a - \frac{M}{M+N}X$, $\quad b - \frac{N}{M+N}X$

$\dfrac{dX}{dt} = k$ (amt chem A)(amt chem B)

factoring gives this form

Example: Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 80 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 10 minutes. How much is formed in 40 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time?

$a = 80g$, $b = 50g$ $M = 2$, $N = 1$

$$\alpha = \frac{80(2+1)}{2} = 120 \qquad \beta = \frac{50(2+1)}{1} = 150$$

$$\frac{dX}{dt} = k(120 - X)(150 - X)$$

$$\frac{dX}{(120-X)(150-X)} = k\,dt$$

PFD:

$$\int \left(\frac{1/30}{120 - X} - \frac{1/30}{150 - X} \right) dX = k\,dt$$

Initial condition: $X(0) = 0$

$$X(t) = \frac{600(e^{0.0018t} - 1)}{5(e^{0.0018t} - 4)}$$